Lec 3 : DSP

$x(n) = \{1, 0.5, 0, -1\}$ ($n=-2$, $n=-1$, $n=0$, $n=1$)

$h(n) = \{0.5, 1, 0, 1\}$ ($n=0$)

Find : $x(n)$ $h(n)$ ← Convolution

Sol

$y(n) = x(n) * h(n)$

$n_{start} = n_{x_{start}} + n_{h_{start}}$ $\quad -2 + 0 = -2$

$n_{end} = n_{x_{end}} + n_{h_{end}} = 1 + 3 = 4$

$$y(n) = \sum_{K=-2}^{1} x(K)\, h(n-K)$$

الباقي تعويض

* Properties of Convolution

① Commutative Property of Convolution

الإبدال

$$y(n) = \sum_{K=-\infty}^{\infty} x(K)\, h(n-K)$$

$$= \sum_{K=-\infty}^{\infty} x(n-K)\, h(K)$$

إثبات بها الرياضي

assume $n-K=m$

$$K=n-m$$

$$y(n)=\sum_{-\infty}^{\infty} x(K)\,h(n-K)$$

$$=\sum_{m=-\infty}^{\infty} x(n-m)\,h(m)$$

$K=-\infty \Rightarrow m\longrightarrow\infty$

$K=\infty \Rightarrow m\longrightarrow -\infty$

$m\longrightarrow K$ استبدال

$$y(n)=\sum_{K=-\infty}^{\infty} x(n-K)\,h(K) \;//$$

② Causality of LTI

$$y(n)=\sum_{K=-\infty}^{\infty} h(K)\,x(n-K)$$

$$=\sum_{K=-\infty}^{-1} h(K)\,x(n-K)$$

$$+\sum_{K=0}^{\infty} h(K)\,x(n-K)$$

$$= \Big[h(-1)\,x(n+1) + h(-2)\,x(n+2) + h(-3)\,x(n+3) \cdots \Big] + \Big[h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) \cdots \Big]$$

$h(n) = 0 \;,\; n < 0$ ← د ه على الجزء السالب

← لو تحققه كده يبقى الجزء الخرج هو اللى معانا

معناها انه لايعتمد على (future value) Causal ← وتكون

→ The system is Causal if:

$$\boxed{h(n) = 0 \;,\; n < 0}$$

EX $h(n) = \{1, -2, \underset{\uparrow}{0}, \frac{1}{2}, 3\} \rightarrow$ (n=0)

* ↳ non-Causal system

because n-values less than 0 is existed

② $h(n) = \{\underset{\uparrow}{1}, -2, 0, \frac{1}{2}, 3\} \rightarrow$ Causal system (n=0)

3 Lec3

③ stability of LTI

$$y(n) = \sum_{K=-\infty}^{\infty} h(K)\, x(n-K)$$

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \longrightarrow \text{stable sys.}$$

* Z-Transform:-

Cont. time system

time domain $\xrightarrow{L.T}$ s-domain

$\xleftarrow{\mathcal{L}^{-1}.T}$

discrete time system

discrete time domain $\xrightarrow{Z.T}$ Z-domain

$\xleftarrow{Z^{-1}.T}$

$$\boxed{Z = e^{-TS}}$$

$T \longrightarrow$ sampling-time.

$$x(n) = \sum_{-\infty}^{\infty} x(K)\, \delta(n-K)$$

if $T \neq 1$

$$x(nT) = \sum_{K=-\infty}^{\infty} x(KT)\, \delta(n-K)T$$

$\Downarrow$

Laplace Transform

$(\mathcal{L}\cdot T)$

$$\mathcal{L}\left[x^*(t)\right] = \mathcal{L}\left[x(nT)\right] = \sum_{K=-\infty}^{\infty} x(KT) \cdot e^{-KTs}$$

$$X^*(s) = \sum_{K=-\infty}^{\infty} x(KT)\, e^{-KTs} \longrightarrow \left(e^{Ts}\right)^{-K}$$

$$Z = e^{Ts}$$

$Z \cdot T$

$$\boxed{X(z) = \sum_{-\infty}^{\infty} x(KT) \cdot z^{-K}}$$

For a discrete time sequence:-

$x(nT) \equiv x(n)\Big|_{T=1\,sec}$ the Z-transform

For this sequence is

$$Z\left[x(n)\right] = X(z) = \sum_{K=-\infty}^{\infty} x(KT)\, e^{-K} \quad T=1\,sec$$

$$= X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Ex Find Z.T for $\delta(n)$

Sol

$$Z\left[\delta(n)\right] = \sum_{n=-\infty}^{\infty} \delta(n)\, z^{-n}$$

For Causal signals and system

$$Z\left[x(n)\right] = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

→ LTI system and Causal system.

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + \cdots$$

$$x(n) = \delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Z\left[x(n) = \delta(n)\right] = 1$$

Ex Find Z.T for $x(n) = u(n)$

Sol

$$X(z) = \sum_{n=0}^{\infty} \underbrace{x(n)}_{u(n)}\, z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + \cdots$$

$$X(z) = \frac{1}{1 - z^{-1}} \qquad z^{-1} < 1$$

$$X(z) = \frac{z}{z-1}$$

→ Region of convergence (RoC)

on Previous example

RoC is $z^{-1} < 1 \Rightarrow z > 1$

EX:2 find Z transform for $x(n) = e^{-an}$

Sol

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-an} z^{-n} = 1 + e^{-a} z^{-1} + e^{-2a} z^{-2} + \cdots$$

$$e^{-a} z^{-1} < 1$$

$$X(z) = \frac{1}{1 - e^{-a} z^{-1}} = \frac{z}{z - e^{-a}}$$

RoC $e^{-a} z^{-1} < 1$

EX 3 $x(n) = n$, find Z.T

$$X(z) = \sum_{n=0}^{\infty} n z^{-n}$$

$$0 + z^{-1} + 2z^{-3} + 3z^{-3} + \cdots \rightarrow (1)$$

$$z^{-1}X(z) = z^{-2} + 2z^{-3} + 3z^{-4} \cdots \quad \text{②}$$

① - ②

$$X(z) - z^{-1}X(z) = z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$X(z)\left[1 - z^{-1}\right] = \frac{z^{-1}}{1 - z^{-1}} \qquad \text{ROC: } z^{-1} < 1$$

$$X(z) = \frac{z^{-1}}{(1 - z^{-1})^2} \qquad * \frac{z^2}{z^2}$$

$$X(z) = \frac{z}{(z-1)^2}$$

EX 4 Find Z.T for $x(n) = a^n$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= 1 + az^{-1} + a^2 z^{-2} + \cdots$$

$$\text{ROC: } az^{-1} < 1$$

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

EX5

$x(n) = \sin(\omega n)$ Find $X(z)$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega n) = \frac{e^{j\omega n} - e^{-j\omega n}}{2j}$$

$$\text{for } x(n) = \sin(\omega n) = \frac{e^{j\omega n} - e^{-j\omega n}}{2j}$$

$$X(z) = \sum_{n=0}^{\infty} \left[\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right] z^{-n}$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} \left[e^{j\omega n} - e^{-j\omega n}\right] z^{-n}$$

$$= \frac{1}{2j} \left[\sum_{0}^{\infty} e^{j\omega n} z^{-n} - \sum_{0}^{\infty} e^{-j\omega n} z^{-n}\right]$$

$$X(z) = \frac{1}{2J}\left[\left(1 + e^{J\omega} z^{-1} + e^{J2\omega} z^{-2} + \cdots\right) - \left(1 + e^{-J\omega} z^{-1} + e^{-J2\omega} z^{-2} + \cdots\right)\right]$$

$$= \frac{1}{2J}\left[\frac{1}{1 - e^{J\omega} z^{-1}} - \frac{1}{1 - e^{-J\omega} z^{-1}}\right]$$

$$= \frac{1}{2J}\left[\frac{(1 - e^{-J\omega} z^{-1}) - (1 - e^{J\omega} z^{-1})}{(1 - e^{J\omega} z^{-1})(1 - e^{-J\omega} z^{-1})}\right]$$

$$= \frac{1}{2J}\left[\frac{(e^{J\omega} - e^{-J\omega}) z^{-1}}{1 - 2\left(\frac{e^{J\omega} + e^{-J\omega}}{2}\right) z^{-1} + z^{-2}}\right]$$

$\frac{e^{J\omega} + e^{-J\omega}}{2} \rightarrow \cos\omega$

$$X(z) = \frac{z^{-1}\left(\frac{e^{J\omega} - e^{-J\omega}}{2J}\right)}{1 - 2\left(\frac{e^{J\omega} + e^{-J\omega}}{2}\right) z^{-1} + z^{-2}}$$

$$= \frac{z^{-1} * \sin\omega}{1 - 2\cos\omega z^{-1} + z^{-2}}$$

[11] Lec 3.

$$X(z) = \frac{Z \sin(w)}{Z^2 - 2Z\cos w + 1}$$

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Z - Transform

$x(n) \xrightarrow{Z\text{-}T} X(z)$

$\delta(n)$	1
$u(n)$	$\frac{Z}{Z-1}$
$e^{\pm an}$	$\frac{Z}{Z - e^{\pm a}}$
a^n	$\frac{Z}{Z-a}$
n	$\frac{Z}{(Z-1)^2}$
$\sin(wn)$	$\frac{Z\sin(w)}{Z^2 - 2\cos(w) + 1}$

[12] Lec3

$$\cos(w) \longrightarrow \frac{z(z-\cos w)}{z^2 - 2z\cos w + 1}$$

* Properties of Z-transform

① $Z[a\, x(n)] = a\, X(z)$

(a: Const.)

Ex: Find Z.T $[3u(n)] = Z.T[3] = 3\frac{z}{z-1}$

② $Z[x_1(n) \pm x_2(n)] = X_1(z) \pm X_2(z)$

③ $Z[e^{\pm an}\, x(n)] = X(z)\Big|_{z = z e^{\mp a}}$

Ex: Find Z.T for $n e^{2n}$

$$Z[n e^{2n}] = \frac{z}{(z-1)^2}\Big|_{z = z e^{-2}}$$

$$= \frac{z e^{-2}}{(z e^{-2} - 1)^2}$$

④ $Z[n\,x(n)] = -z\,\frac{d\,X(z)}{dz}$

Find Z.T n^2

$Z[n^2] = Z[n * n] = -z\,\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$

($x(n)$ & $x(n)$)

$= -z\,\frac{(z-1)^2 * 1 - z(2)(z-1)}{(z-1)^4}$

$= -z\,\frac{(z-1) - 2z}{(z-1)^3} = -z\,\frac{-z-1}{(z-1)^3}$

$= \frac{z(z+1)}{(z-1)^3}$

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⑤ $Z[a^n\,x(n)] = X(z)\Big|_{z=\frac{z}{a}}$

EX Z.T for $[n\,a^n]$

$= \frac{z}{(z-1)^2}\Big|_{z=\frac{z}{a}}$

$$= \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2} = \frac{az}{(z-a)^2}$$

Ex $x(n) = a^n e^{2n}$ Find $X(z)$

$$e^{2n} \xrightarrow{z\cdot T} \frac{z}{z-e^2}$$

$$a^n e^{2n} \xrightarrow{z\cdot T} \left.\frac{z}{z-e^2}\right|_{z=\frac{z}{a}}$$

$$= \frac{\frac{z}{a}}{\frac{z}{a}-e^2} = \boxed{\frac{z}{z-ae^2}}$$

another solution

$$a^n \xrightarrow{z\cdot T} \frac{z}{z-a}$$

$$e^{2n} a^n \xrightarrow{z\cdot T} \left.\frac{z}{z-a}\right|_{z=e^{-2}*z}$$

$$= \frac{ze^{-2}}{ze^{-2}-a} = \boxed{\frac{z}{z-ae^2}}$$

6 Convolution in z-domain

$$x_1(n) * x_2(n) \xrightarrow{Z.T} X_1(z) \cdot X_2(z)$$

Ex $x_1(n) = 3\delta(n) + 2\delta(n-1)$

$x_2(n) = 2\delta(n) - \delta(n-2)$

Find $X(z) = Z[x_1(n) * x_2(n)]$

$$x(z) = Z[x_1(n) * x_2(n)] = X_1(z) * X_2(z)$$

$$Z[x(n-m)] = z^{-m} X(z) \rightarrow Z.I.C$$

zero initial condition

$$X_1(z) = 3 + 2z^{-1}$$

$$X_2(z) = 2 - z^{-2}$$

$$X(z) = X_1(z) * X_2(z) = (3+2z^{-1})(2-z^{-2})$$

Ex

$x(n) = \{1, 0, 1, 0.5, 2\}$

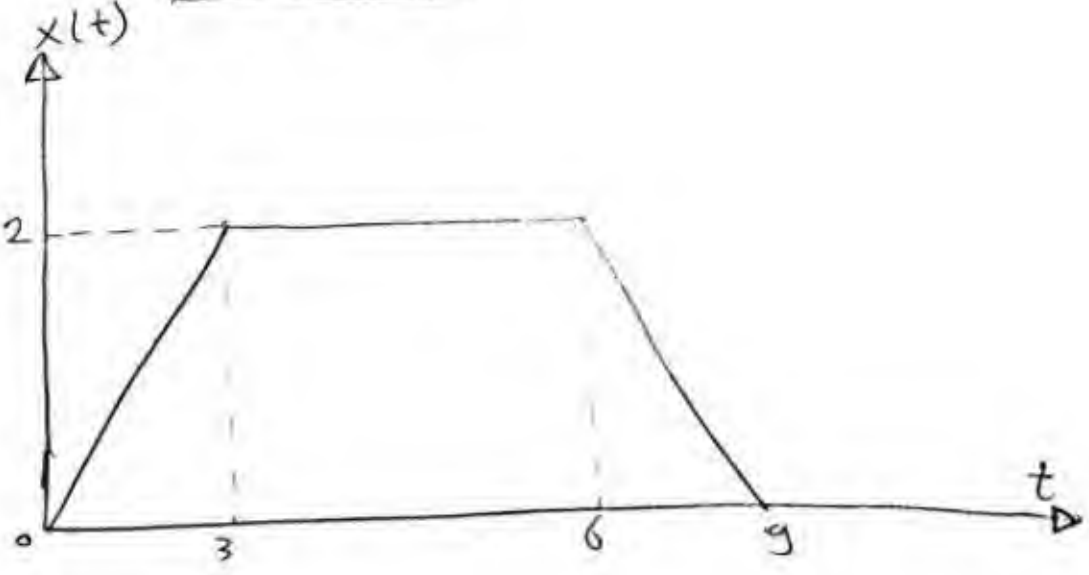
Find $X(z)$

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} \cdots$$

$$= 1 + z^{-2} + 0.5 z^{-3} + 2 z^{-4}$$

Report



→ Find $X(z)$